§0. Lagrangians/Hamiltonians in classical mechanics and geometric quantization

Let N be an n-dimensional smooth manifold and denote by TN its tangent bundle.

Definition:

Let $\mathcal{L}(t, x, \xi)$ be a "Lagrangian" defined over TN with $x = (x_1, \dots, x_n) \in N$, $\xi = (\xi_1, \dots, \xi_n) \in TN$, and $\gamma : [a,b] \longrightarrow N$ a smooth curve on N

Then

$$S = \int_a^b \mathcal{L}(t, \gamma(t), \gamma'(t))\,dt$$

is called the "action integral".

A critical point of the action satisfies

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi_j} = \frac{\partial \mathcal{L}}{\partial x_j}, \quad j = 1, \dots, n$$

"Euler-Lagrange equations"

Example:

$\mathcal{L} = \frac{1}{2} m \sum_{j=1}^{n} \xi_j^2 - V$, where V is a potential, gives back Newton's equations for a particle.

Next, let M be a smooth manifold of dimension $2n$. A "symplectic form" ω is a non-degenerate, closed 2-form on M. ($\omega^n \neq 0$ on M)

Definition:

A smooth manifold equipped with a symplectic form is called a "symplectic manifold".

Given a smooth vector field X on M, the correspondence $X \mapsto \iota(X)\omega$ gives isomorphism between 1-forms on $M \longleftrightarrow$ smooth vector fields on M

Definition (Hamiltonian vector field):

Let f be a smooth function on M.

Define vector field X_f by

$$\iota(X_f)\,\omega = df$$

X_f is "Hamiltonian vector field" for f.

Definition (Poisson bracket):

For smooth functions f and g on M, define

$$\{f, g\} = -\omega(X_f, X_g)$$

Properties:

- bilinear and anti-symmetric
- satisfy Jacobi identity:

  $$\{f, \{g,h\}\} + \{g, \{h,f\}\} + \{h, \{f,g\}\} = 0$$

- $\{f, gh\} = \{f,g\}h + g\{f,h\}$ (derivation)

We have

$$\omega(X_f, X_g) = \iota(X_f)\omega(X_g) = df(X_g) = X_g f = -X_f g$$

Denote by $\mathcal{X}(M)$ the space of smooth vector fields on M. Then

$$[X, Y]f = X(Yf) - Y(Xf), \quad f \in C^\infty(M),$$

$\longrightarrow$ $\mathcal{X}(M)$ is equipped with a Lie algebra structure.

$C^\infty(M)$ is a Lie algebra by the Poisson bracket and we have the following relation:

$$[X_f, X_g] = X_{\{f,g\}}$$

$\longrightarrow$ $f \mapsto X_f$ defines a Lie-algebra homomorphism

Consider a smooth function H on M called the "Hamiltonian" and the associated vector field X_H. Then

$$\frac{df}{dt} = X_H f = \{H, f\} \Longrightarrow \frac{dH}{dt} = \{H, H\} = 0$$

Complex line bundles and quantization

Let M be a smooth manifold and E a smooth vector bundle on M.

Denote

- $T^*M_{\mathbb{C}}$: complexification of T^*M
- $\Gamma(E)$: space of smooth sections of E
- $\Gamma(T^*M_{\mathbb{C}} \otimes E)$: space of smooth sections of $T^*M_{\mathbb{C}} \otimes E$

Definition (connection):

A "connection" on E is a $\mathbb{C}$-linear map

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M_{\mathbb{C}} \otimes E)$$

such that the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

holds for $f \in C^\infty(f)$ and $s \in \Gamma(E)$.

Definition (covariant derivative):

For a vector field $X \in \Gamma(T^*M_{\mathbb{C}})$ define a linear map $\nabla_X : \Gamma(E) \to \Gamma(E)$ by

$$(\nabla_X s)(x) = (\nabla s)(X_x).$$

Definition (complex line bundle):

A "complex line bundle" is a complex vector bundle of rank 1.

Let L be a complex line bundle with Hermitian metric over M.

open covering: $M = \bigcup_j U_j \to L = U_j \times \mathbb{C}$ over U_j

(trivialization of L)

Let ∇ be a connection on L given by

$$\nabla = d - 2\pi\sqrt{-1}\,\alpha_j \quad \text{on } U_j,$$

where α_j is 1-form on U_j.

Definition (first Chern class):

$d\alpha_j$ defines a global 2-form on M, the "first Chern-form" of ∇, denoted by $c_1(\nabla)$.

Its de Rham cohomology class $[c_1(\nabla)] \in H^2(M, \mathbb{R})$ is called "first Chern class" of L.

We have

$$[c_1(\nabla)] \in \operatorname{Im} \iota,$$

where $\iota : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$

is the inclusion map

- classical Hamiltonian mechanics :

$$\frac{df}{dt} = \{H, f\}$$

- quantum mechanics :

$$\frac{d\hat{f}}{dt} = [\hat{H}, \hat{f}], \quad \text{where } \hat{f} = \sqrt{-1}\,\tilde{f}/\hbar$$

and $\tilde{f}$ is the operator version of f

"Heisenberg equation"

Let us apply this program of quantization to a symplectic manifold (M, ω) together with a complex line bundle L :

- choose L and ∇ such that

$$c_1(\nabla) = \omega$$

- Denote by $\Gamma(M, L)$ the space of smooth sections of L with inner product

$$\langle s_1, s_2 \rangle = \int_M (s_1(x), s_2(x)) \frac{\omega^n}{n!} \quad (*)$$

where $(s_1(x), s_2(x))$ is hermitian metric on L

- Denote by $\mathcal{H}$ the space of L^2 sections of L with respect to $(*)$.

$\longrightarrow$ Hilbert space

- quantization map: for $f \in C^\infty(M)$ define $f \longmapsto \hat{f}$ by $\hat{f}s = \nabla_{X_f} s - 2\pi\sqrt{-1}\, f s$, $s \in \mathcal{H}$

Using $\omega(X_f, X_g) = -\{f, g\}$, one can verify

$$[\hat{f}, \hat{g}] = \widehat{\{f, g\}}$$

$\longrightarrow$ $f \mapsto \hat{f}$ determines a representation of $C^\infty(M)$ on $\mathcal{H}$ as a Lie algebra.

Example:

$N = \mathbb{R}^n \longrightarrow M = T^*N$ with symplectic form

$$\omega = \sum_{j=1}^{n} dp_j \wedge dq_j$$

coordinates on N

coordinates on TN

Define a connection on the trivial line bundle $L = M \times \mathbb{C}$ by:

$$\nabla = d - \sqrt{-1}\,\hbar^{-1}\theta, \quad \theta = \sum_{j=1}^{n} p_j\, dq_j$$

For a section $s: M \longrightarrow L$ define

$$\tilde{f}\, s = \sqrt{-1}\,\hbar \nabla_{X_f} s + f s$$

$$\longrightarrow \tilde{p}_j = -\sqrt{-1}\,\hbar \frac{\partial}{\partial q_j}, \quad \tilde{q}_j = \sqrt{-1}\,\hbar \frac{\partial}{\partial p_j} + q_j$$

Let $\mathcal{H}_0$ be the subspace of $\Gamma(M, L)$ consisting of L^2 sections depending *only* on $q_1, \dots, q_n \longrightarrow$ For $s \in \mathcal{H}_0$: $\tilde{p}_j s = -\sqrt{-1}\,\hbar \frac{\partial}{\partial q_j} s$, $\tilde{q}_j s = q_j s$

$\rightarrow$ recovered canonical quantization in quantum mechanics

Definition (Polarization):

Let (M, ω) be a symplectic manifold of dimension $2n$ and $TM_{\mathbb{C}}$ the complexified tangent bundle.

$V_P \subset TM_{\mathbb{C}}$ subbundle is "integrable" if: for $X, Y : M \rightarrow V_P \Rightarrow [X, Y] : M \rightarrow V_P$

V_P is "Lagrangian" if

$$\forall x \in M : \dim (V_P)_x = n \quad \text{and}$$
$$\omega|_{(V_P)_x} = 0$$

A Lagrangian V_P is called "polarization", if it is integrable.

Define $\mathcal{H}_P = \{ s \in \mathcal{H} \mid \nabla_X s = 0, X \in \Gamma(M, V_P) \}$

$\uparrow$

quantum Hilbert space

Definition (Kähler polarization):

Let (M, ω) be a Kähler manifold, set $V_P = TM^{(0,1)}$

$\rightarrow \mathcal{H}_P = H^0(M, L)$ space of holomorphic sections